Introduction to Haskell Programming

Data structures – Binary search trees

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Creating a binary tree

- Create a binary tree from a list

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- As **balanced** as possible

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- **Strategy**:

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- As **balanced** as possible
- **Strategy**:
  - Split the list in two halves
  - Recursively create a binary tree from each half
  - Join them together

Creating a binary tree

- Creating a balanced tree from a list

```
createTree :: [a] → BTree a
createTree [] = Nil
createTree xs = Node tl y tr where
    m               = length xs `div` 2
    (tl, y:ys)      = (createTree (take m xs), drop m xs)
    tr              = createTree ys

levels (createTree [0..14]) = [7,3,11,1,5,9,13,0,2,4,6,8,10,12,14]
height (createTree [0..14]) = 4
```

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  - Problem remains even if we use `splitAt`
  - Can we avoid that?

Deriving a more efficient `createTree`

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Deriving a more efficient `createTree`

- Avoid **`length`** in recursive calls
- New version of `createTree`

```
createTree :: [a] → BTree a
createTree xs        = ct (length xs) xs where
    ct 0 xs          = Nil
    ct n xs          = Node tl y tr where
        m            = n `div` 2
        (tl, y:ys)   = (ct m (take m xs), drop m xs)
        tr           = ct (n-m-1) ys
```

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- Let `go n xs = (ct n (take n xs), drop n xs)`

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```

- Let `go n xs = (ct n (take n xs), drop n xs)`
- Then `createTree xs = fst (go (length xs) xs)`

Deriving a more efficient `createTree`

- Calculating a direct definition of `go`, assuming `0 < n ⩽ length xs`

```
go n xs
= (ct n (take n xs), drop n xs)
= (Node tl y tr,     drop n xs) where
    m            = n `div` 2
    (tl, y:rest) = (ct m (take m (take n xs)), drop m (take n xs))
    tr           = ct (n-m-1) rest
```

Deriving a more efficient `createTree`

- If `m < n`, and `drop m xs = y:ys`, then ...

```
take m (take n xs) = take m xs

y:rest  = drop m (take n xs) = take (n-m) (drop m xs)
        = take (n-m) (y:ys) = y: take (n-m-1) ys
So rest = take (n-m-1) ys

drop n xs   = drop (n-m) (drop m xs)
            = drop (n-m) (y:ys)
            = drop (n-m-1) ys
```

Deriving a more efficient `createTree`

- Substituting appropriately ...

```
go n xs

= (Node tl y tr, drop (n-m-1) ys) where
    m           = n `div` 2
    (tl, y:ys)  = (ct m (take m xs), drop m xs)
    tr          = ct (n-m-1) (take (n-m-1) ys)

= (Node tl y tr, zs) where
    m           = n `div` 2
    (tl, y:ys)  = go m xs
    (tr, zs)    = go (n-m-1) ys
```

Creating a binary tree

- Creating a tree in linear time

```
createTree :: [a] → BTree a
createTree l = fst (go (length l) l) where
    go :: Int → [a] → (BTree a, [a])
    go 0 xs = (Nil, xs)
    go n xs = (Node tl y tr, zs) where
        m          = n `div` 2
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```

- $T(n) = 2T(n/2) + c$ and hence $T(n) = (2n - 1)c$

The `Set` ADT

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  - `search` – check whether a given value is an element of the set

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  - `insertInto` – insert a given value into the set
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  - `fromList` – turn a list into a set
- Straightforward implementation

```
module Set(Set, insertInto, deleteFrom,
               search, fromList) where
data Set a = Set [a]
```

The Set ADT

```
search :: Eq a ⇒ a → Set a → Bool
search x (Set xs) = x `elem` xs

insertInto :: Eq a ⇒ a → Set a → Set a
insertInto x (Set xs) = Set (if x `elem` xs
                                then xs else x:xs)

deleteFrom :: Eq a ⇒ a → Set a → Set a
deleteFrom x (Set xs) = Set (filter (/=x) xs)

fromList :: [a] → Set a
fromList = Set . nub
```

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- `insertInto` takes $O(n)$ time
- `deleteFrom` takes $O(n)$ time
- A sequence of n operations takes $O(n^2)$ time
- We can do better if the elements admit an order

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- In a binary search tree:
  - values in the left subtree are smaller than the root
  - values in the right subtree are larger than the root

Binary search trees

- The binary search tree

```
Node tl x tr
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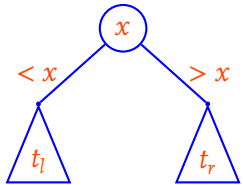
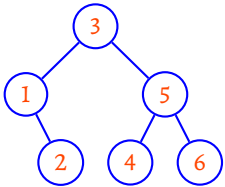
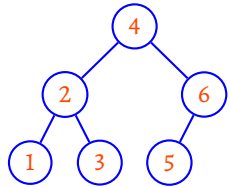
- Pictorially ...



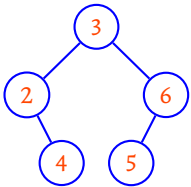
Binary search trees

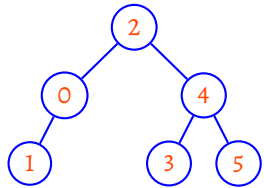
- Examples





- Non-examples





BST

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- Is a tree empty?

```
isEmpty :: BST a → Bool
isEmpty Nil = True
isEmpty _   = False
```

Is it a search tree?

- Just naming it BST does not make it a binary search tree

```
isBST :: Ord a ⇒ BST a → Bool
isBST = isSorted . inorder where
    isSorted [] = True
    isSorted xs = and (zipWith (⩽) xs (tail xs))
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- Recall `inorder`

```
inorder :: BST a → [a]
inorder t                  = go t [] where
    go Nil xs              = xs
    go (Node tl x tr) xs   = go tl (x:go tr xs)
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- If `t` is a BST, `inorder t` is a sorted list

Minimum in a tree

- `mint` gives the minimum value in a non-empty tree:

```
mint :: Ord a ⇒ BST a → a
mint (Node tl x tr)  = min x (min y z)
where   y = if isEmpty tl then x else mint tl
        z = if isEmpty tr then x else mint tr
```

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- `maxt` is similar (uses **`max`** instead of **`min`**)

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Searching in a tree

- Haskell code transcribes the search strategy:

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searchBST :: Ord a ⇒ a → BST a → Bool
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- **Worst case running time**: Height of the tree
- `@` allows one to refer to a data value by a name
- From list to binary search tree ...and back to list

```
fromList = foldr insertBST emptyBST
treeSort = inorder . fromList
```

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```
deleteMax :: Ord a ⇒ BST a → (a, BST a)
deleteMax (Node tl x Nil) = (x, tl)
deleteMax (Node tl x tr) = let (y, ty) = deleteMax tr
                           in (y, Node tl x ty)
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    - If left subtree is empty, just slide the right subtree up a level
    - If left subtree is non-empty, replace root by maximum element of left subtree
    - Search tree property is preserved

Deleting from a tree

- Code for delete:

```
deleteBST :: Ord a ⇒ a → BST a → BST a
deleteBST v Nil = Nil
deleteBST v (Node tl x tr)
| v < x          = Node (deleteBST v tl) x tr
| v > x          = Node tl x (deleteBST v tr)
| v == x         = if isEmpty tl then tr
                                 else Node ty y tr
    where (y, ty) = deleteMax tl
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- Inserting in ascending or descending order results in highly skewed trees