Introduction to Haskell Programming

Data structures – AVL trees

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Complexity of binary search tree operations

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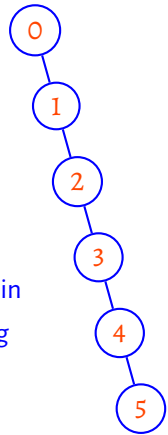
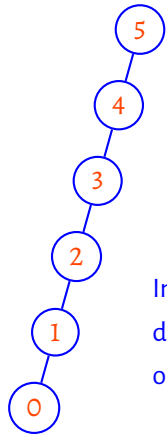
- In general, a tree might not be balanced
- Inserting in ascending or descending order results in highly skewed trees

Balance



Inserting in ascending order



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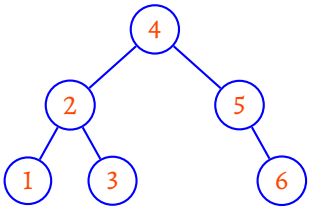
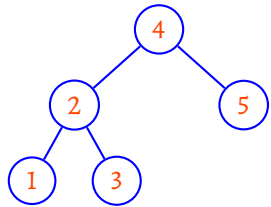


Height-balanced
and size-balanced



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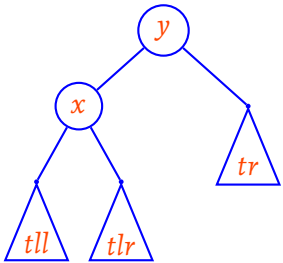
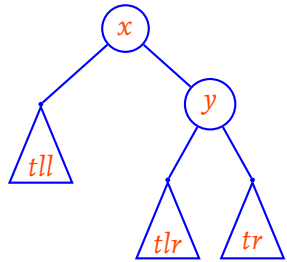
  - If its height is h, then $n \geqslant S(h) \geqslant 2^{\lfloor h/2 \rfloor}$
  - So $(h-1)/2 \leqslant \lfloor h/2 \rfloor \leqslant \log n$, and hence $h \leqslant 2 \log n + 1$

Tree rotations – rotate right





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- In Haskell:

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rotateRight (Node (Node tll x tlr) y tr)
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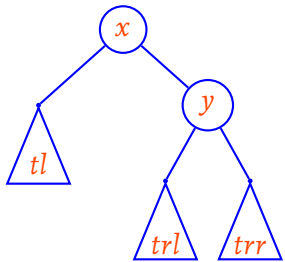

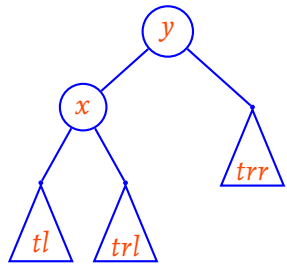

- Useful when *trr* has large height
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rotateLeft (Node tl x (Node trl y trr))
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- Also need a measure of how skewed a tree is – its **slope**

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slope :: AVL a → Int
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slope (Node _ tl _ tr) = height tl - height tr
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AVL trees – rotates

- Since we maintain height at each node, we need to adjust it after each operation:

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rotateRight :: AVL a → AVL a
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- Violations happen only at nodes visited by operation
- We rebalance each node on the path visited by operation

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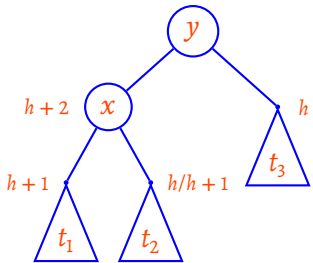

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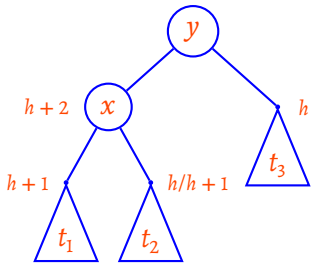

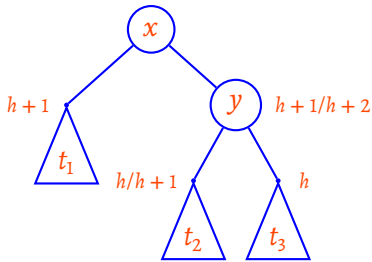

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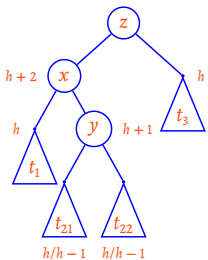

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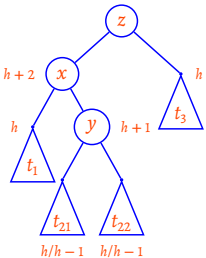

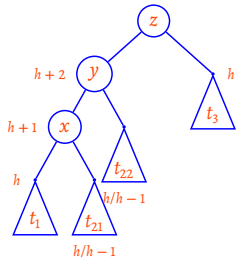

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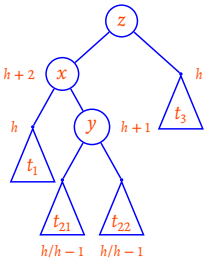

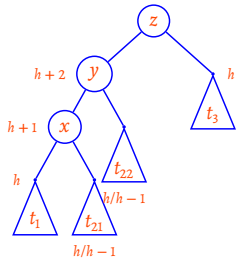

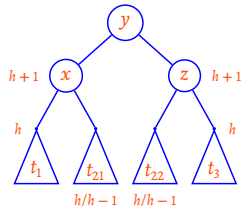

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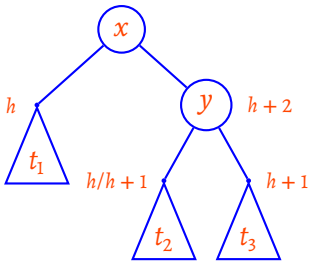

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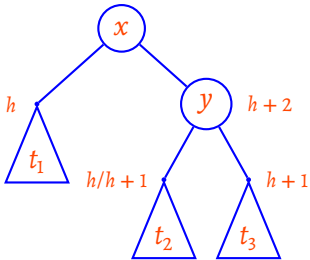

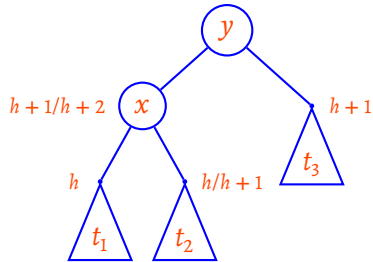

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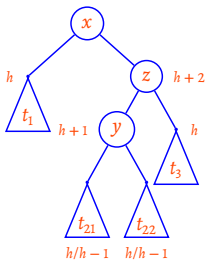

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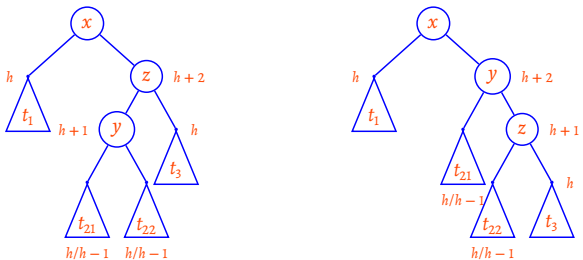

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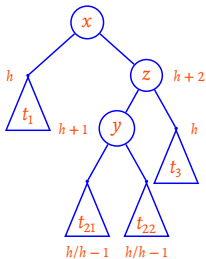

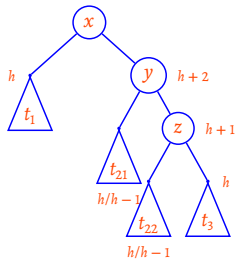

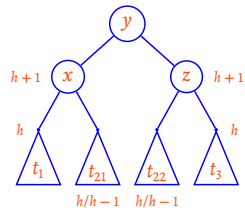

Rebalancing in Haskell

- The `rebalance` function

```
rebalance :: AVL a → AVL a
rebalance t@(Node h tl x tr) = case (slope t, slope tl, slope tr) of
        (2, -1, _)  → rotateRight (Node h (rotateLeft tl) x tr)
        (2, _, _)   → rotateRight t
        (-2, _, 1)  → rotateLeft (Node h tl x (rotateRight tr))
        (-2, _, _)  → rotateLeft t
        _           → t
```

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```

- **Constant time** operation

Searching in an AVL tree

```
searchAVL :: Ord a ⇒ a → AVL a → Bool
searchAVL v Nil                 = False
searchAVL v (Node _ tl x tr)    = case compare v x of
    LT  → searchAVL v tl
    GT  → searchAVL v tr
    EQ  → True
```

Inserting in a tree

```
insertAVL :: Ord a ⇒ a → AVL a → AVL a
insertAVL v Nil                 = Node 1 Nil v Nil
insertAVL v t@(Node h tl x tr)  = let
    (ntl, ntr)      = (insertAVL v tl, insertAVL v tr)
    nhl             = 1 + max (height ntl) (height tr)
    nhr             = 1 + max (height tl) (height ntr)
        in case compare v x of
        LT  → rebalance (Node nhl ntl x tr)
        GT  → rebalance (Node nhr tl x ntr)
        EQ  → t
```

Deleting the maximum element

```
deleteMax :: AVL a → (a,AVL a)
deleteMax (Node _ tl x Nil) = (x, tl)
deleteMax (Node h tl x tr)  = (y, rebalance (Node nh tl x ty)) where
    (y,ty)                  = deleteMax tr
    nh                      = 1 + max (height tl) (height ty)
```

Deleting from a tree

```
deleteAVL :: Ord a ⇒ a → AVL a → AVL a
deleteAVL v Nil          = Nil
deleteAVL v t@(Node h tl x tr)  = let
    (y,ty)      = deleteMax tl
    (ntl, ntr)  = (deleteAVL v tl, deleteAVL v tr)
    nhy         = 1 + max (height ty) (height tr)
    nhl         = 1 + max (height ntl) (height tr)
    nhr         = 1 + max (height tl) (height ntr)
        in case compare v x of
        LT  → rebalance (Node nhl ntl x tr)
        GT  → rebalance (Node nhr tl x ntr)
        EQ  → if isEmpty tl then tr
                else rebalance (Node nhy ty y tr)
```

Tree operations – complexity

- Left and right rotates take constant time

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- Thus each operation takes $O(\log n)$ time
- A sequence of n operations take at most $O(n \log n)$ time

Other useful functions

- Create an empty AVL tree:

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```

- Create an AVL tree from a list ($O(n \log n)$ time):

```
createAVL :: Ord a ⇒ [a] → AVL a
createAVL = foldr insertAVL emptyAVL
```

Other useful functions

- Create a sorted list from an AVL tree:

```
inorder :: Ord a ⇒ AVL a → [a]
inorder t                    = t +: [] where
    Nil             +:  xs   = xs
    Node _ tl x tr  +:  xs   = tl +: (x:(tr +: xs))
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- $T(n) = T(m) + T(n - m - 1) + c$, so $T(n) = cn = O(n)$
- We can sort a list in $O(n \log n)$ time by:

```
treesort :: Ord a ⇒ [a] → [a]
treesort = inorder . createAVL
```

Other useful functions

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- `t` will not be a search tree in general
- If `l` itself is sorted, `t` is search tree
- This is just the smart `createTree` we saw in a previous class
- Works in $O(n)$ time

inorderTree

```
inorderTree :: [a] → AVL a
inorderTree l = fst (go (length l) l) where
    go :: Int → [a] → (AVL a, [a])
    go 0 xs = (Nil, xs)
    go n xs = (Node h tl y tr, zs) where
        m    = n `div` 2
        (tl, y:ys)  = go m xs
        (tr, zs)    = go (n-m-1) ys
        h           = 1 + max (height tl) (height tr)
```

A module for AVL trees

- Saved in `AVL.hs`

```
module AVL(AVL, emptyAVL, isEmpty,
        insertAVL, deleteAVL, searchAVL,
        createAVL, inorder, inorderTree) where

data AVL a = Nil | Node Int (AVL a) a (AVL a)
    deriving Eq

instance Show a ⇒ Show (AVL a) where
    show t = ...
```

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```

- Can be used to define the `Set` ADT

The Set ADT again

```
module Set(Set, emptySet, createSet,
        insertInto, deleteFrom, search,
        setUnion, setIntersect, setDiff) where
import AVL

data Set a = Set (AVL a)
instance (Ord a, Show a) ⇒ Show (Set a) where
    show (Set t) = show (inorder t)

emptySet :: Ord a ⇒ Set a
emptySet = Set emptyAVL
```

The Set ADT again

```
createSet :: Ord a ⇒ [a] → Set a
createSet = Set . createAVL

search :: Ord a ⇒ a → Set a → Bool
search x (Set t) = searchAVL x t

insertInto :: Ord a ⇒ a → Set a → Set a
insertInto x (Set t) = Set (insertAVL x t)

deleteFrom :: Ord a ⇒ a → Set a → Set a
deleteFrom x (Set t) = Set (deleteAVL x t)
```

More set operations

```
setUnion :: Ord a ⇒ Set a → Set a → Set a
setUnion (Set t1) (Set t2) = Set $ inorderTree $
            unionMerge (inorder t1) (inorder t2)

unionMerge :: Ord a ⇒ [a] → [a] → [a]
unionMerge [] ys = ys
unionMerge xs [] = xs
unionMerge (x:xs) (y:ys)
    | x < y  = x:unionMerge xs (y:ys)
    | y < x  = y:unionMerge (x:xs) ys
    | x == y = x:unionMerge xs ys
```

More set operations

```
setIntersect :: Ord a ⇒ Set a → Set a → Set a
setIntersect (Set t1) (Set t2) = Set $ inorderTree $
        intersectMerge (inorder t1) (inorder t2)

intersectMerge :: Ord a ⇒ [a] → [a] → [a]
intersectMerge [] ys = []
intersectMerge xs [] = []
intersectMerge (x:xs) (y:ys)
    | x < y  = intersectMerge xs (y:ys)
    | y < x  = intersectMerge (x:xs) ys
    | x == y = x:intersectMerge xs ys
```

More set operations

```
setDiff :: Ord a ⇒ Set a → Set a → Set a
setDiff (Set t1) (Set t2) = Set $ inorderTree $
            diffMerge (inorder t1) (inorder t2)

diffMerge :: Ord a ⇒ [a] → [a] → [a]
diffMerge [] ys = []
diffMerge xs [] = xs
diffMerge (x:xs) (y:ys)
    | x < y  = x:diffMerge xs (y:ys)
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    | x == y = diffMerge xs ys
```

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- Allows us to easily specify complex transformations on data