$W_1, W_2, \dots, W_n$ subspaces of V

"$\sum W_i$" = "$W_1 + W_2 + \cdots + W_n$" = span of $\bigcup_i W_i = \{W_1 + W_2 + \cdots + W_n \mid w_i \in W_i\}$

We say V is an internal direct sum of W_i if any of the following equiv. cond. is true

(II) $W_1, \dots, W_n$ are lin. indep and span V. (recall $\{W_i\}$ indep means $(W_1 + \cdots + W_n = 0 \quad w_i \in W_i)$ $\Downarrow$ each $w_i = 0$)

$\Updownarrow$ I and II are equivalent. III is just a fact.

(I) $\underbrace{W_1 \times W_2 \times \cdots \times W_n}_{\text{V-space via slotwise op}^n} \xrightarrow{s} V$ map given by $s(W_1, W_2, \dots, W_n) = W_1 + \cdots + W_n$ is an isomorphism

(Rmk. s automatically linear, surj. if $\sum W_i = V$, injective iff $\{W_i\}$ are indep.

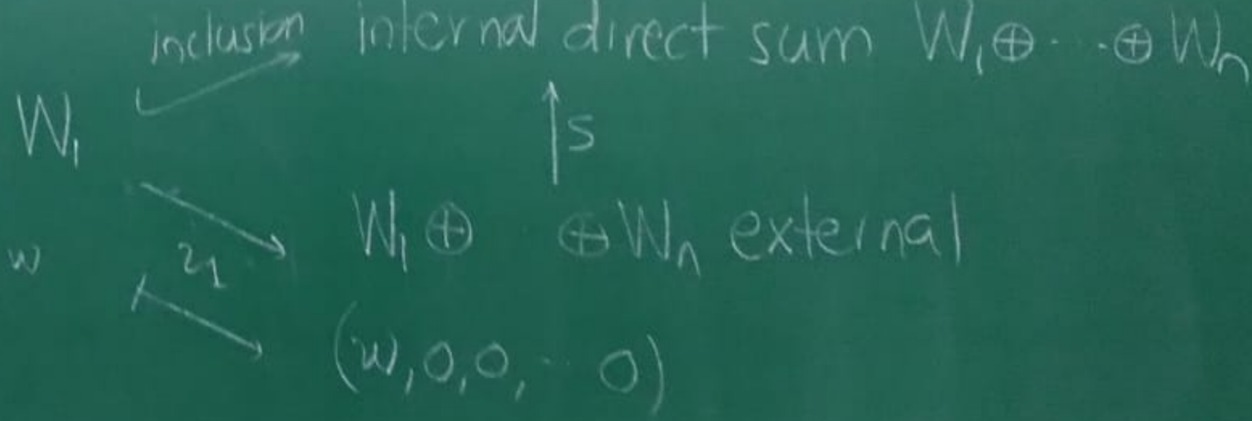
(III) Univ property { Given any vspace U and linear maps $f_i : W_i \to U$ there exists a unique linear $f : W_1 \times \cdots \times W_n \to U$ s.t. $f(0, \dots, 0, w_i, 0 \cdots 0) = f_i(w_i)$

Notation We denote direct sum by $\oplus$ $W_1 \oplus \cdots \oplus W_n$ or $\bigoplus_{i=1}^{n} W_i$

[Q What happens when we have an infinite family?]

Rmk We have linear maps

Language



Rmk If $\{W_\alpha\}_{\alpha \in I}$ is any family of subspaces of V, then $\bigcap_\alpha W_\alpha$ is also a subspace of V.

Spl case $n=2$ When $V = W_1 \oplus W_2$ we say "W_2 is a complement of W_1"

· Thm If W_1, W_2 are subspaces of V then

$$\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

Pf 1 Take a basis $\{a_1, \ldots, a_k\}$ of $W_1 \cap W_2$. Extend to get a basis of W_1, also to get one of W_2 etc.

Wrt $(V_1, V_2, V_3) = B$ matrix of f is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

How to find V_1, V_2, V_3 from A

We will see!

Pf 2 $W_1 \oplus W_2 \xrightarrow{S} V$ external

$\ker(S) = \{(w, -w) \mid w \in W_1 \cap W_2\} \cong W_1 \cap W_2$

Example looking ahead

$f(v) = Av$ where

$$A = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix}$$

Calculate $f(v)$ for

$$V_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad V_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$f(V_1) = V_1 \quad f(V_2) = V_2 \quad f(V_3) = -V_3$

$\text{span}(V_1, V_2) = \text{plane } 2x - y - z = 0$

Linear algebra 0 is valid for vector spaces over any field

(Informal)

↳ A field is a set with well behaved 4 arithmetic operations

Def^n A field F consists of a set F with two binary operations add^n & mult^n

s.t. ① $(F, +)$ is an abelian gp with neutral elt 0.

② $(F^\times, \cdot)$ (where $F^\times = F \setminus \{0\}$) is an abelian gp with neutral elt 1

③ Distributivity holds. $a(b+c) = ab+ac$

Silly stuff

Show $0 \cdot a = 0$ etc

$(w, 0, 0, \cdots 0)$

End of class

Examples of fields: $\mathbb{R}, \mathbb{C}, \mathbb{Q}$

$\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$

$\mathbb{R}(x) = \left\{ \frac{p(x)}{q(x)} \mid p, q \text{ polynomials} \right\}$

$\mathbb{Q}[i] = \{a + bi\} \subset \mathbb{C}$

$\mathbb{F}_p$ = integers mod a prime p