Analysis of operator T on f.dim V-space V over field F

λ is an evalue of $T \iff \ker(\lambda I - T) \neq 0 \iff \det(\lambda I - T) = 0$ i.e. λ a root of char poly

Recipe to test diagonalizability of T

① Find char poly $P_T(x) = \det(xI - T)$. Find its roots

② For each root λ, find a basis for $\ker(\lambda I - T)$

T diag $\iff$ V has an eigenbasis corresp to T

$\iff$ { ① $P_T(x)$ factors fully AND

② $V = \bigoplus_{\lambda \in F} \ker(\lambda I - T)$ ⇕ $\dim V = \sum_{\lambda \in F} \dim \ker(\lambda I - T)$ Uses indep of espaces

T can fail to be diagonal for two reasons.

(i) "not enough evalues" i.e. $P_T(x)$ does not factor fully (defect of F)

(ii) "not enough evectors for a given evalue" $\dim V > \sum$ dim of espaces

For now we will see how to tackle situation (ii), assuming $P_T(x)$ factors fully

Agenda from yesterday T operator on V of dim n

(A) T nilpotent and diagonal $\Rightarrow T=0$

(B) T nilpotent $\Leftrightarrow$ T has a triangular matrix with diag all 0 $\Leftrightarrow P_T(x)=x^n \Leftrightarrow T^n=0$
(1) (2)

(C) $P_T(x)$ factors fully $\Leftrightarrow$ T has a Δ matrix with diag entries roots of $P_T(x)$

(D) Fitting decomposition of V relative to T

$V = \ker T^\infty \oplus \operatorname{Im} T^\infty$

∞ = sufficiently large N ($N = \dim V$ works!)

Proofs (C) $\Leftarrow$ If T has matrix $\begin{bmatrix} a_{11} & * \\ 0 & a_{nn} \end{bmatrix}$ wrt some basis then $P_T(x) = \prod_{i=1}^{n} (x-a_{ii})$

$\Rightarrow$: Suppose $P_T(x)=(x-\lambda_1)\cdots(x-\lambda_n)$

How to choose a basis to get a Δ matrix? Let v_1 be a λ_1-evector.

$$A=\left[\begin{array}{c|c} \lambda_1 & * \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array}\right]$$ (v_1) for any completion of basis (pick one) matrix of T looks like this

We want a matrix P s.t. $P^{-1}AP$ is upper Δ

$P_T(A) = (x-\lambda_1) \cdots (x-\lambda_n) = (x-\lambda_1) P_T(B) \Rightarrow P_T(B) = (x-\lambda_2) \cdots (x-\lambda_n)$

b/c $xI - A = \left[\begin{array}{c|c} x-\lambda_1 & * \\ \hline 0 & xI_{n-1} - B \end{array}\right]$

B is the matrix of an operator on $\operatorname{span}(v_2, \cdots, v_n)$

By induction, on n we have an $(n-1)\times(n-1)$ invertible matrix Q s.t. $Q^{-1}BQ$ is upper Δ

Define $P = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q \end{array}\right]$ Then $P^{-1} = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q^{-1} \end{array}\right]$

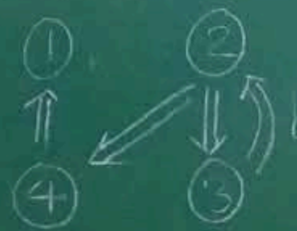
$$P^{-1}AP = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q^{-1} \end{array}\right]\left[\begin{array}{c|c} \lambda_1 & * \\ \hline 0 & B \end{array}\right]\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q \end{array}\right] = \left[\begin{array}{c|c} \lambda_1 & * \\ \hline 0 & Q^{-1}B \end{array}\right]\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & Q \end{array}\right] = \left[\begin{array}{c|c} \lambda_1 & *Q \\ \hline 0 & Q^{-1}BQ \end{array}\right] !$$

Rmk Can also analyze decreasing chain

$V \supset T(V) \supset T^2(V) \supset \cdots$ (terminates in $T^{\infty}V = 0$ if T is nilpotent)

and prove ① ⇒ ② for nilpotent T

Pf of B



Left to do: ① ⇒ ② Pf For any operator T on V observe: Increasing chain of subspaces

$0 \subset \ker T \subset \ker(T^2) \subset \cdots$ (If T nilpotent eventually we get $\ker(T^N) = V$)

$v \in \ker(T^k) \Leftrightarrow T^k v = 0$

$v \in \ker(T^{k+1}) \Leftrightarrow T^{k+1} v = 0$

If $\ker(T^k) = \ker(T^{k+1})$ then $\ker(T^k) = \ker(T^{k+j})$ $\forall j \geq 0$.

Write a precise pf by following your nose.

If T is nilpotent, $\ker(T^{\text{large}}) = V$ As $\dim \ker(T^i)$ have to grow at each $i \to i+1$ and must reach n, # steps needed to reach n is $\leq n$

T nilp $\Rightarrow T^{\dim V} = 0$ So ① ⇒ ④

Take basis of $\ker T$

Add to get basis of $\ker(T^2)$

Continue ...

Matrix of $\ker(T^{\infty})$ wrt this basis is strictly upper Δ.