For an operator $T$ on v-space $V$, a subspace $W$ of $V$ is called $T$-stable ($T$-invariant) if $T(W) \subset W$.

i.e "$T|_W$ = restriction of $T$ to domain $W$" is an operator on $W$.

Let $\underbrace{w_1, \cdots, w_k}_{\text{basis of } W}, \underbrace{u_1, \cdots, u_\ell}_{\text{call span of these } U}$ be a basis of $V$

$\therefore V = W \oplus U$.

Matrix of $T$ wrt to this basis looks like

| | $w_1$ ⋯ $w_k$ | $u_1$ ⋯ $u_\ell$ |
|---|---|---|
| $w_1$ ⋮ $w_k$ | $A$ | $B$ |
| $u_1$ ⋮ $u_\ell$ | $O$ | $D$ |

Rmk 1-dim'l $T$-inv subspace ⇝ an eigenvector

key question

Can I make $B = O$ using a suitable basis?

If that is possible then $U$ is also $T$-invariant and we can say

"$T = T|_W \oplus T_U$"

T an operator on f.dim V. space V of dim n over field F

**Thm (Fitting decomposition)** ① $\ker T \subset \ker T^2 \subset \cdots$
$\operatorname{Im} T \supset \operatorname{Im} T^2 \supset \cdots$
} Each chain stabilizes, as soon as a term repeats. no later than $T^n$ term

- $V = \ker(T^\infty) \oplus \operatorname{im}(T^\infty)$
- $\ker(T^\infty)$ is the largest T-stable subspace on which T is nilpotent
- $\operatorname{im}(T^\infty)$ is the **largest** T-stable subspace on which T is an isom

Notation $\ker(T^\infty) = \ker T^n$
$\operatorname{im}(T^\infty) = \operatorname{im} T^n$

Obeserve that $\ker(T^i)$, $\operatorname{im}(T^i)$ are T-stable

Ex. Write full pf. Show spanning explicitly. Show $T|_{\operatorname{Im}(T^\infty)}$ is bijective

**Proof of ⊕** By rank nullity applied to $T^n$, we know
$\dim(\ker T^\infty) + \dim(\operatorname{im} T^\infty) = \dim V$. So E.T.S $\ker(T^n) \cap \operatorname{im}(T^n) = 0$
Let $v \in$ intersection $T^n v = 0$ and $T^n w = v \Rightarrow T^n v = T^{2n} w = 0$ ....

Assume char poly $P_T(x) = \prod_{i=1}^{k} (x-\lambda_i)^{m_i}$ Note: $\sum_{i=1}^{k} m_i = n$ Ex: Why is the factorization unique?

Recall: $T$ is diag $\iff V = \bigoplus_j \ker(\lambda_j I - T)$

$\iff \dim \ker(\lambda_j I - T) = m_j \ \forall j$

Ex: $\dim \ker(\lambda_j I - T) \leq m_j$

"geometric mult $\leq$ algebraic multiplicity"

**Thm** If $P_T(x) = \prod_j (x-\lambda_j)^{m_j}$, $V = \bigoplus_j \ker(\lambda_j I - T)^{\infty}$

"can take $\infty = \dim V$"

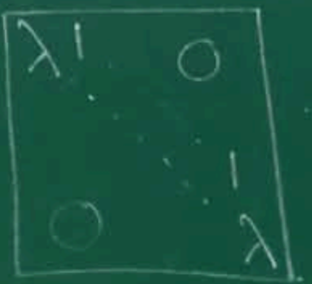
Ex: Can we take $\infty = m_j$ (for $\lambda_j$)?

Def$^{n}$ $\ker(\lambda I - T)^{\infty}$ = generalized $\lambda$-eigenspace of $T$. Observe that gen $\lambda$-espaces of $T$ are $T$-stable

Pf: You do, using: We have long division for polynomials [optional]

just like in $\mathbb{Z}$. As a result we have gcd of polynomials $P_1, P_2$ which can be obtained as $f_1P_1 + f_2P_2$

**Jordan canonical form** If $P_T(x) = \prod (x-\lambda_j)^{m_j}$ then each generalized eigenspace $\ker (\lambda I - T)^\infty$ has a basis wrt which the matrix of $T$ has a block diagonal form with each of these, smaller blocks

of Jordan type $J_{\lambda,d} =$ 

$d \times d$

Moreover the sizes of these J-blocks are uniquely determined by $T$.

Just understand the statement. We'll do examples in class. Proof: later (maybe) and optional