Grade breakup

1. In-sem 45
2. End-sem 45
3. Best of (1) & (2) 10

In-sem. best 45 of following

Test 1	15
Test 2	15
Quizzes	15 = 5×3
HW	5

Drop lowest of quizzes/HW

or

Reduce wt of one test to 10

Meet me tonight (wait for email) / tomorrow morning after 8:30 am

Classes. Tomorrow at 2pm. Remaining 6 classes optional → next W Th Fr

last week Tu W Th

F.dim inner product space V over $F = \mathbb{C}$ (later $\mathbb{R}$) Notation $M^* = \overline{M}^T$

↓ has orthonormal bases. (For an ON basis B, let $[v]$ = coord vector of v)

Recall $\langle v, w \rangle = [v]^* [w]$ by calculation

Def^n Let T be any linear operator on V. Let $M = \left\{ \begin{array}{l} \text{matrix of } T \text{ wrt an} \\ \text{ON basis } B \end{array} \right.$

T^* = "Adjoint of T" is the operator whose matrix wrt the same ON basis is M^*

Q: Does this depend on choice of B? Let's check. Choose another ON basis $B' = BP$

with P unitary ie $P^*P = I$ ie cols of P are ON in $\mathbb{C}^n$. Matrix of T wrt B' is P^*MP

Enough to check $(P^* M P)^* \overset{?}{=}$ Matrix of already defined T^* wrt new basis

$\downarrow$ $\qquad\qquad$ $=$

Yes, b/c $(AB)^* = B^* A^*$ and $(A^*)^* = A$ $\qquad$ $P^* M^* P$

Key equation $\langle Tv, w\rangle = [Tv]^*[w] = (M[v])^*[w] = [v]^* M^* [w] = [v]^* [T^* w] = \langle v, T^* w\rangle$

Ex. If an operator S satisfies $\langle Tv, w\rangle = \langle v, Sw\rangle$ then $S = T^*$

by def^n of T^*!

Rmk As $\langle T^* v, w\rangle = \langle v, T^{**} w\rangle = \langle v, Tw\rangle$ as $T^{**} = T$

Note: this is why symmetric/Hermitian operators are called self-adjoint

$\langle v, T^* w - Sw\rangle = 0$

$\forall\, v, w$

Take $v = T^* w - Sw$

Nice operators on inner product space (IPS) V / $F = \mathbb{C}$ or $\mathbb{R}$ (For $\mathbb{R}$, $^* =$ transpose)

	Unitary/orthogonal	Hermitian/Symmetric	Normal/real normal
1) eq^n wrt $\langle , \rangle$	$\langle Tv, Tw\rangle = \langle v, w\rangle$	$\langle Tv, w\rangle = \langle v, Tw\rangle$	$\langle Tv, Tw\rangle = \langle T^*v, T^*w\rangle$
2) rel^n with T^*	$T^*T = I$	$T = T^*$	$T^*T = TT^*$
3) matrix M wrt ON basis	$M^*M = I$ (cols are O.N. and so are rows)	$M = M^*$ (matrix is Hermitian/symmetric)	$M^*M = MM^*$
2) $\Rightarrow$ 1):	$\langle Tv, Tw\rangle = \langle v, T^*Tw\rangle \overset{(2)}{=} \langle v, w\rangle$	$\langle Tv, w\rangle = \langle v, T^*w\rangle = \langle v, Tw\rangle$	$\langle Tv, Tw\rangle = \langle v, T^*Tw\rangle = \langle v, TT^*w\rangle = \langle T^*v, T^*w\rangle$
1) $\Rightarrow$ 2)	use $v = T^*Tw - w$	use $v = Tw - T^*w$	

Rmk Unitary/Hermitian ⇒ Normal

Spectral thms over $\mathbb{C}$

- T normal $\iff$ V has an O.N. basis consisting of evectors of T
- T unitary $\iff$ " " with each evalue λ s.t. $|\lambda|=1$
- T Hermitian $\iff$ " " " λ real

Pf. ($\Leftarrow$ is easy) For $\Rightarrow$ Take evector v with say $Tv=\lambda v$. Let $W=\mathbb{C}v$. $\quad$ v O.N. basis of $W^\perp$

Key claim: $W^\perp$ is T-invariant. (This implies $T|_{W^\perp}$ is normal e.g. by looking at its matrix (check)

Then induction on $\dim V$ finishes the pf.

Claim $W^\perp$ is $\bar{T}$-inv.

Pf. let $u \in W^\perp \iff \langle v, u\rangle = 0$

Want $Tu \in W^\perp \iff \langle v, Tu\rangle = 0$

We know $\langle v, Tu\rangle = \langle T^* v, u\rangle \overset{!}{=} \langle \bar{\lambda} v, u\rangle = \lambda \langle v, u\rangle = 0$

(We will show in fact v is an evector for T^* too, with evalue $\bar{\lambda}$)

→ Exercise/tomorrow