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BSC202C02

Solution to B1 continued

1.

(a) $A \cdot t = \begin{pmatrix} 1\cdot t + t\cdot 1 \\ 1\cdot t + 1\cdot 1 + 1\cdot t \\ 2\cdot(1-t) \end{pmatrix} = 2\cdot\begin{pmatrix} t \\ 1 \\ 1-t \end{pmatrix}$

Hence t is an eigenvector.

(b). Consider $\lambda I - A_t$:

$$\begin{pmatrix} \lambda-1 & t & 0 \\ 1 & \lambda-1 & 1 \\ 0 & 0 & \lambda-2 \end{pmatrix} = A_{t,11}A_{t,22}A_{t,33} + (-1)A_{t,12}A_{t,21}A_{t,33}$$

$$= (\lambda-1)(\lambda-1)(\lambda-2) - t\cdot 1\cdot(\lambda-2)$$

$$= (\lambda-2)\left[(\lambda-1)^2 - t\right]$$

Note that 2 is always an eigenvalue.

$(\lambda-1)^2 - t = 0 \Rightarrow (\lambda-1)^2 = t$

Thus, $t \geq 0$. If $t = 0$, we have a double root, so we must check $\dim(\ker(A_0 - I))$.

Otherwise, ~~dim~~ $t > 0$ gives us two roots. ~~Since the eigenvectors correspond~~ We must check for $\lambda = 2$.

There, we get, $(\lambda-1)^2 = 1 \Rightarrow 1 - t = 0$. Hence, for $t = 1$, we would have two repeated roots again. Otherwise, we get three distinct eigenvalues. Hence, A_t is definitely diagonalisable for $t \in \mathbb{R}_{\geq 0} \setminus \{1\}$. Let us check for $t = 0, 1$.

Solution to B1 continued in pages 12 to 28? Write the page number here ____

B2. Write your solution to B2 below.

- For $t = 0$, we have $\lambda = 2, 1$. $A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

$\dim \ker(A_0 - 2I) = \dim \ker \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. This gives you,

$x \in \ker(A_0 - 2I) \Rightarrow (A_0 - 2I)x = 0$

$\Rightarrow (-1)x_1 = 0, \quad x_1 + (-1)x_2 + x_3 = 0$

$\Rightarrow x_2 = x_3$.

So, we have,

$\ker(A_0 - 2I) = \text{span}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, which has dim. 1.

$\dim\left(\ker\left(A_0 - I\right)\right) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Again, it is easy to see that $\ker(A_0 - I) = \text{span}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ which has dim. 1. Hence A_0 is not diagonalisable.

- For $t=1$, we have roots, $\lambda = 2, 0$, where 2 is a double root.

$\dim(\ker(A_1 - 0I)) = \dim(\ker(A_1))$.

$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ let us convert it to a RREF.

$A_{1,\text{RREF}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. $\Rightarrow \ker(A_1) = \text{span}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

so it has dim. 1

- $\lambda = 2$

Solution to B2 continued

$$A_1 - 2I = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

its RREF is, $(A_1 - 2I)_{RREF} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, which gives you

$$\mathrm{Ker}(A_1 - 2I) = \mathrm{span}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Hence the dimensions do not add upto 3, so ~~they're~~ its not diagonalisable.

For $t \in \mathbb{R}_{>0} / \{1\}$, A_t is diagonilasble, since it has three distinct eigenvalues and since eigenvectors corresponding to different eigenvalues are independent (shown in class), $f(x) = Ax$ in $f: \mathbb{R}^3 \to \mathbb{R}^3$ an operator, and invariance of dimension (so that three lin independent vectors form a basis) give us an eigenbasis, which diagonalise A_t.

(c). Let $t \notin \mathbb{R}_{>0} / \{1\}$. Suppose there exists a basis, v_1, v_2, v_3 that diagonalises A_t. Then,

$$A_t = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \Rightarrow A_t(v_i) = \lambda_i v_i .$$

$\Rightarrow v_i \in \mathrm{Ker}(A_t - \lambda_i I)$. For $t < 0$, this implies $v_i \in \mathrm{Ker}(A_t - 2I)$

[For others $\lambda \in \mathbb{R}$, $\det(A_t - \lambda I) \neq 0 \Rightarrow A_t - \lambda I$ is invertible $\Rightarrow v_i = 0$ so they don't form a basis].

Solution to B2 continued in pages 12 to 28? Write the page number here ____

But, ~~$v_1 \in \mathrm{Ker}(A_t - 2I)$, v_2~~ as we showed in class,

alg. multiplicity $\geq$ geom. multiplicity $\Rightarrow \dim(\mathrm{Ker}(A_t - 2I)) \leq 1$.

so they cannot be linearly ind. We have already worked out the dimensions for $t = 0$, $t = 1$ □

B3. Write your solution to B3 below.

2. (a) $p_A(x) = \det(xI - A) = \det\begin{pmatrix} x & 1 \\ 1 & x-1 \end{pmatrix}$

$= x(x-1) - 1$

$= x^2 - x - 1$

(b). $\mathbb{F}_2$. The only elements of it are 0, 1. Clearly neither have det $= 0$. So it is not diagonalisable.

In fact. $x^2 - x - 1 = 0 \Rightarrow x = \frac{1 \pm \sqrt{5}}{2}$. if $\exists x \in \mathbb{F}_p : x^2 = 5$, then $\mathbb{F}_p$ won't have it diagonlizable.

Indeed, it is obvious that if $\exists x: x^2 = 5$, it will work. and there will be distinct solutions as $\frac{1 \pm x}{2} = \frac{1-x}{2} \Rightarrow x + x = 0$ but $\mathbb{F}_p$ has characteristic p, so this implies $x = 0$ for $p > 2$. (Also $1/2$ exists for $p > 2$ since $\mathbb{F}_p$ is a field, $2 \in \mathbb{F}_p$].

So $x^2 = 5 \Rightarrow 5 = 0$, which is only true for $p = 5$.

Anyway, if $\exists\, x: x^2 - x - 1$, we claim $\exists y: y^2 = 5$ in $\mathbb{F}_p$. evident from the choice $y = 2x - 1$.

Thus it suffices to check for $\sqrt{5}$. But since $2^2 = (p-2)^2$, we only need to check the first half for squares.

• $\mathbb{F}_3$. 0 works. Thus, $y = \pm\frac{1}{2} = 3$.

~~We have to check $\dim(\ker(A - 3I))$.~~

If is a double root, indeed, $x^2 - x - 1 = (x-3)^2$.

So we have to check $\dim(\ker(A - 3I))$, because we only know $\dim(\ker(A - 3I)) \geq 1$. Whether it is 1 or 2 is to be decided. (Easy to see that it won't be 2 though)

B4. Write your solution to B4 below. (5)

- F11. Note that 4, 7 are roots of $\sqrt{5}$. Thus indeed we do have two eigen values, $\frac{1+4}{2} = 8$, $\frac{1-4}{2} = 4$.
  Again by independence of eigenvectors having these eigen values, we ~~know~~ have that A is diagonalisable.

Solution to B3 continued

3.

(a) No, they cannot span. As we showed in class,

$$\dim(A+B) = \dim(A) + \dim(B) - \dim(A \cap B) \leq \dim(A) + \dim(B) = 9.$$

Hence, span of A and B = A+B has dimension at most 9, less than dimension of V, so they cannot span V by invariance of dimension.

Yes, they can be independent. Consider a basis $v_1, \ldots, v_{10}$ of V. Then,

$$A = \text{span}\{v_1, v_2\} \text{ and } B = \text{span}\{v_3, \ldots, v_9\}$$

are independent, since v_i form a basis.

(b) $f(a,b) = 0 \Rightarrow a+b = 0 \Rightarrow a = -b$.

since $a \in A \Rightarrow b = -a \in A$ as well.

$\Rightarrow b \in A \cap B$.

Indeed, we see that $f(a,b) = 0 \Rightarrow a = -b \in A \cap B$.

For any $\omega \in A \cap B$, if, $f(\omega, -\omega) = \omega + -\omega = 0$

$\Rightarrow A \cap B = \ker(f)$.

(c)

$\dim(A+B)$	$\dim(A \cap B)$
9	0
8	1
7	2
7	

Solution to B3 continued in pages 12 to 28? Write the page number here _____

(7)

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Evident from the relation we proved in class (or an application of the 2nd isomorphism theorem), and that $\dim(A \cap B) \leq \dim(A)$. This is because ~~if~~ ~~dim~~ there is an injective map $f: (A \cap B) \to A$, the identity. Or well, You note that a lin independent, spanning set of A is certainly ~~not~~ spans of $A \cap B$, so the dimension must be $\leq \dim(A)$.

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4.

(a). Consider a basis of $\ker(T)$ and extend it to a basis of V. Note that T in that basis would've a matrix of the form:

$$A = \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & \\ \vdots & \vdots & \vdots & B \end{pmatrix}$$

[Note that we use invariance of $p_T(x)$ which was shown in class.]

[* the v_i's after v_4 have no entry in the v_i slot because $v_i = a_1 v_1 + a_2 v_2 + a_3 v_3 + \sum a_j v_j$ $\Rightarrow T(v_i) = \sum a_i T(v_j)$ since $v_1, v_2, v_3 \in \ker(T)$]

In particular this gives you

$$p_T(x) = \det(xI - A) = x^3 \cdot \det(xI - B) \to p_B(x) \Rightarrow x^3 \mid p_T(x)$$

because the upper triangular entries being zero ensure that the det expansion is like this.

($\sigma(1)=1$, $\sigma(2)=2$, $\sigma(3)=3$ is fixed. Rest others are 0 is the det of B because otherwise you're just permuting two entries (and $\sigma(i)$ for $i \geq 3$, must be 4 or greater, since the rest of the terms are zero).

By the same logic, if λ is an eigenvalue of T, $(x-\lambda)^{\dim(\ker(T-\lambda I))}$ divides $p_T(x)$

(b). Consider $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that

$T(e_1) = 0$, $T(e_2) = 0$, $T(e_3) = 0$, $T(e_4) = e_4$, $T(e_5) = e_4$.

Then, $\dim(\ker(T)) = 3$, as

$T(v) = T(a_1 e_1 + a_2 e_2 + \cdots + a_5 e_5) = 0 \quad a_4 e_5 + (a_4 + a_5) e_4 = 0$

$\Rightarrow a_4 T(e_4) + a_5 T(e_5) = 0 \Rightarrow$ ~~$(a_4 + a_5) e_5 + a_4 e_4 = 0$~~

$\Rightarrow a_5 = a_4 = 0$.

(9)

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T wrt standard basis is:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\det(xI - A) = \det\begin{pmatrix} x & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & x-1 & 1 \\ 0 & 0 & 0 & 1 & x \end{pmatrix}$$

$$= x^4 \cdot (x-1).$$

This

(c) No. From (a) we note that if $p_T(x) = (x-\lambda_1)^{a_1} \cdots (x-\lambda_k)^{a_k}$

Then $a_{\lambda_i} \geq \dim(\ker(A - \lambda_i I))$.

Note, however that since T is diagonalisable,

$$\sum \dim(\ker(A - \lambda_i I)) = \dim V.$$

Thus $\sum a_i \geq \dim(V)$. But if $a_i > \dim(\ker(A - \lambda_i I))$, this would be a strict inequality. However, note that that the highest power of x in $p_T(x) \leq \dim V$, because T forms a $\dim V \times \dim V$ matrix and x only appears in the diagonal. Thus, highest power of x in $p_T(x)$

$= \sum a_i = \dim V$ forces that $a_i = \dim(\ker(A - \lambda_i I))$

for every i.

(10)

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(5)

(a) Note that v_1 can be any $v \in V$ aside from 0.
There, we have $p^n - 1$ choices. For v_2, it can be anything that is not a scalar multiple of v_1, hence we have $p^n - p$ choices (since p scalars).

Similarly, $v_{i+1} \notin \text{span}\{v_1, \ldots, v_i\}$

$\Rightarrow v_{i+1} \neq c_1 v_1 + \cdots + c_i v_i$ for $c_i \in \mathbb{F}_p$.

Since v_i are lin independent, there are exactly p^i such numbers. Hence we have $p^n - p^i$ choices for v^i.

Therefore we have a total of $(p^n - 1)(p^n - p) \cdots (p^n - p^{k-1})$ many such sequences.

(b) Exactly the same! Any k-dimensional subspace has a basis comprising of k-vectors. Since these must be lin. independent, they're one of the k-length independent ordered sequences.

For the other way around, $\text{span}(v_1, \ldots, v_k)$ would be a k-dimensional subspace of $\mathbb{F}_p^n$
(since v_i span it and v_i are lin. independent, they form a basis of length k).

Hence there are $(p_n - 1) \cdots (p_n - p^{k-1})$ such subspaces.