Quiz 2

Each question is worth 3 to 4 points. Your score is total of best 5.

1. Find $|GL_3(\mathbb{F}_5)|$, i.e., the number of invertible 3×3 matrices with entries in $\mathbb{F}_5$, the field consisting of integers mod 5. You may leave your answer as a product of positive integers (each expressed in ordinary base 10 representation), but be sure to explain your reasoning.

2. Suppose A and B are subspaces of a vector space V. The dimensions of A, B and V are 22, 10, and 25 respectively.

(a) Is it possible for the sum $A + B$ to be direct? If yes, give an example. If not, state why.

(b) Let $A \oplus B$ be the external direct sum of A and B, i.e., $A \oplus B =$ the set $A \times B$ with addition and scalar multiplication defined slotwise. The map $f : A \oplus B \to V$ defined by $f(a, b) = a + b$ is linear. (Check this in your mind if you wish. It is not part of the question.) What is the kernel of this map?

(c) Make a table with two columns as follows. In the first column list all possible values of the dimension of the subspace $A + B$ of V. For each entry in the first column, write on the same line in the second column all possible corresponding values of the dimension of $A \cap B$. Explain.

3. Let $M = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 1 & 3 \end{bmatrix}$. Solve this problem without using any polynomial or variables.

(a) Find a basis of each of the following starting with the definition. 1-eigenspace of M, 3-eigenspace of M, 5-eigenspace of M.

(b) Either show why M cannot be diagonalized OR diagonalize M explicitly, i.e., write a suitable equation involving M, a diagonal matrix D and possibly one or more additional matrices, where all matrices must have explicit constants as entries (no unknowns).

4. Solve the following independent questions.

(a) Find the characteristic polynomial of the matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(b) The characteristic polynomial of a 4×4 matrix M factors fully and has two distinct roots. Write one such matrix M for which the sum of dimensions of its eigenspaces has the minimum possible value. Explain briefly.

5. An operator T on a vector space of dimension 100 satisfies the following. For each positive integer $k \leq 100$ the dimension of the kernel of T^k is k. What is the rank of T? Show how we can get a basis of V with respect to which the matrix of T has the minimum possible number of nonzero entries. Justify.

6. Let $p(x) = (x-22)(x-10)(x-25)$. Suppose an operator S on a vector space W satisfies $p(x)$ but satisfies no proper factor of $p(x)$. Show that each of $22, 10, 25$ must be an eigenvalue of S. (To think about afterwards: must S be diagonalizable?)