Test 2

- Explain clearly. *Do not use the Jordan form unless the question mentions it.* Otherwise you may use any definition/theorem we did in class (clearly specify what result you are using), unless the question asks you to prove something standard. Ask me in case of any doubt.
- Upload your answers on moodle. *Separate pdf/jpeg file for each problem.*

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0. For $M = \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}$ and a positive integer n, let $M^n = \begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix}$.

Using standard linear algebra, find explicit formulas for the four functions a, b, c, d. Your formulas must be in terms of only arithmetic operations on scalars and not involve any matrices. (I know that you can get formulas using ad hoc algebra and prove them using induction. Such a method will get 0 credit. You must show how to arrive at the formulas conceptually using linear algebra.)

1. Let $A = \begin{bmatrix} a & p & x \\ b & 10 & y \\ c & r & z \end{bmatrix}$ and let e_1, e_2, e_3 denote the standard basis of $\mathbb{R}^3$. It is given that e_3 is an eigenvector of A with eigenvalue 11 and $e_2 + 25e_3$ is also an eigenvector of A.

(a) Find all entries of A that are uniquely decided.

(b) Suppose also that A is *noninvertible* (only in this part). Show that A must be diagonalizable.

(c) Give an example of a *nondiagonalizable* matrix A that satisfies the given conditions and has the minimum possible number of nonzero entries.

2. Let S the set of all matrices with entries in $\mathbb{F}_7$ whose characteristic polynomial is $x^5 - 2x^4 + x^3$.

(a) Is there a nilpotent matrix in S?

(b) Find all possible matrices in Jordan form that are in S (order of blocks is immaterial). Must every matrix in S be similar to one in this list?

(c) Write a matrix in S each of whose eigenspaces is one dimensional. Must all such matrices (those in S each of whose eigenspaces is one dimensional) be similar to each other?

3. Let $U = \{[x, y, z]^T | 2x + 3y - 6z = 0\}$. Note that $u = [0, 2, 1]^T$ is in the subspace U of $\mathbb{R}^3$.

(a) Find an orthogonal matrix with column vectors cu, v, w in $\mathbb{R}^3$ such that u, v is a basis of U. Write clearly so the reader can easily distinguish between your u and v. I suggest you avoid cursive.

(b) Let R_θ denote anticlockwise rotation of $\mathbb{R}^3$ around the axis $\mathbb{R}u$ by angle θ. Find a matrix M such that $R_\theta([a, b, c]^T) = M[a, b, c]^T$ for any a, b, c. (The equation has function application on the LHS and matrix multiplication on the RHS.) Your answer M may be written as a product of matrices but each of these matrices must have explicitly specified entries.

(c) Find scalars p, q, r such that $[1, 1, 1]^T = pu + qv + rw$. What is the orthogonal projection of $[1, 1, 1]^T$ on U?

4. Let w be a unit vector in a Euclidean space E with inner product $\langle , \rangle$.

(a) For an arbitrary vector v in E, show to write $v = aw + z$ where z is perpendicular to w. This is standard, but you must derive the formula, not just state it.

(b) Let R_w denote the reflection in hyperplane $w^\perp$ consisting of vectors perpendicular to w. In other words R_w send w to $-w$ and keeps each vector in $w^\perp$ fixed. Write a formula for $R_w(v)$ in terms of w, v and the inner product.

(c) Calculate the reflection of $[1, 1, 1]^T$ in the plane defined by $2x + 3y - 6z = 0$ in $\mathbb{R}^3$ (same as U in the previous problem).

5. Solve the following short independent questions.

A) Is there a 2×2 matrix over the field $\mathbb{R}$ that is NOT similar to an upper triangular matrix?

B) Let $T(v) = Mv$ for the matrix $M = \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}$ from Q0. Is there an orthogonal basis of $\mathbb{R}^2$ with respect to which T has an upper triangular matrix? If not, justify. If yes, find such a basis.

C) Maximize $2x + 3y + 6z$ for points (x, y, z) on the unit sphere. Find all (x, y, z) where the maximum is attained. (Hint: Cauchy-Schwarz)

D) Let T be an operator on a vector space V. It is easy to see the following (do not prove this):

(T is diagonalizable) $\implies$ (Every vector in V is a sum of eigenvectors of T).

Is the converse true? Prove/give a counterexample.

E) Specify all subsets S of $\mathbb{C}$ such that $S =$ the set of eigenvalues of some orthogonal matrix P. (So the entries of P are real but multiplication by P is regarded as an operator on some $\mathbb{C}^n$.)

F) Let $H =$ the space of $n \times n$ complex matrices. It is easy to see that $\langle A, B \rangle = \text{trace}(A^*B)$ makes H into an inner product space. Find an orthonormal basis for H.