Problem set 1 , January 30, 2020

Let R be a ring with unity.

1. Let $R = \left\{ . \begin{pmatrix} a & \overline{b} \\ 0 & c \end{pmatrix} \right\}$, where $a, c \in \mathbb{Z}$ and $b \in \mathbb{Z}/p\mathbb{Z}$ where p is a prime number.
   (a) Show that R is a ring. Is it commutative.
   (b) Let $\alpha = \left\{ \begin{pmatrix} a & \overline{p} \\ 0 & 1 \end{pmatrix} \right\}$. Determine a so that α is a left zero divisor. Is it a right zero divisor.
   (c) Let $\beta = \left\{ \begin{pmatrix} 0 & \overline{1} \\ 0 & 0 \end{pmatrix} \right\}$. Determine whether β is a left /right zero divisor. Is there an integer n so that $\beta^n = 0$.
2. Let $i^2 = j^2 = k^s = -1$ and $ij = -ji = k$.
   (a) Let $\mathbb{H} = \mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot i \oplus \mathbb{R} \cdot j \oplus \mathbb{R} \cdot k$. Show that $\mathbb{H}$ is a division ring. Determine the center of $\mathbb{H}$.
   (b) Let $R = \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot i \oplus \mathbb{Z} \cdot j \oplus \mathbb{Z} \cdot k$
   Is R is a division ring. Determine the center of R. What are the units in R.
3. Consider the ring $\mathbb{C}[x; \sigma]$ where σ denotes complex conjugation on $\mathbb{C}$ and for any $a \in \mathbb{C}$, $xa = \sigma(a)x$.
   (a) Define $(\sum a_i x^i)(\sum b_j x^j) = \sum a_j \sigma^i(b_j) x^{i+j}$ and show that $\mathbb{C}[x; \sigma]$ is a ring.
   (b) Show that the center of R is $\mathbb{R}[x^2]$
   (c) Show that $R/R \cdot (x^2 + 1)$ is isomorphic to the division ring of real quartenions.
   (d) Let $\phi : R \to M_2(\mathbb{C})$ be given by $\phi(x) = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}$ and $\phi(r) = \begin{pmatrix} r & 0 \\ 0 & \sigma(r) \end{pmatrix}$. Show that $\ker(\phi) = (x^4 + 1)$
4. Let D be a division ring with center R. Let $D[x]$ be the polynomial ring over D.
   (a) Show that the center of $D[x]$ is $R[x]$.
   (b) Let $a \in D \setminus R$. Show that the ideal generated by $x - a$ in $D[x]$ is the unit ideal.
   (c) Show that any ideal in $I \subset D[x]$ is of the form $D[x] \cdot (h)$ where $h \in R[x]$.