Tutorial 4- Monday 22/9/2025

We will assume that all rings are nonzero, are commutative and have the unit element.

1. Let $= \mathbb{C}[x, y]$. Show that $(xy - 3y^2, x^2 + 4y^2 - 1, 13y^3 - y) = (y, x - 1) \cap (13y^2 - 1, x - 3y) \cap (y, x + 1)$. Which of the ideals $(y, x - 1)$, $(13y^2 - 1, x - 3y)$, $(y, x + 1)$ are maximal in R.
2. Let F be a field. Prove that the ring $F[x, x^{-1}]$ of Laruent polynomials is a PID.
3. Let $R = \mathbb{Z}[\omega]$. Let $p \neq 3$ be a prime.
   (a) Show that the polynomial $x^2 + x + 1$ has a root if and only if $p \cong 1(mod\ 4)$,
   (b) (p) is maximal ideal if and only if $p \cong 1(mod\ 3)$
   (c) p factors in R if and only if $p = a^2 + ab + b^2$ for some integers a and b.
4. Let f, g be polynomials in $\mathbb{C}[x, y]$ with no common factor. Show that the ring $R = \mathbb{C}[x, y]/(f, g)$ is a finite dimensional vector space over $\mathbb{C}$.
5. Let R be the ring of polynomials in $cos(t)$ and $sin(t)$, with real coefficients.
   (a) Show that $R \cong \mathbb{R}[x, y]/(x^2 + y^2 - 1)$.
   (b) Show that R is not a UFD.
   (c) Prove that $S = \mathbb{C}[x, y]/(x^2 + y^2 - 1)$ is a PID.
   (d) Determine the units in R and S.